DIFFERENTIAL TOPOLOGY MID-TERM EXAM, 2014

Q.1 Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$. Prove that $\mathbb{S}^1 \times \mathbb{S}^1$ is a manifold.

Ans. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the map $g(x, y) = x^2 + y^2$. Here $dg_{(x,y)} = 2xdx + 2ydy$. Then, $dg_{(x,y)}$ is surjective for all $(x, y) \neq \{(0, 0)\}$. Then, 1 is a regular value of g and so by Preimage Theorem, $\mathbb{S}^1 = g^{-1}(1)$ is a submanifold of $\mathbb{R}^2$, hence is a manifold. Thus, $\mathbb{S}^1 \times \mathbb{S}^1$ is also a manifold.

Q.2 If X is compact and Y is connected. Show that every submersion $f : X \to Y$ is surjective.

Ans. By Local Submersion Theorem, we can observe that if $y \in f(X)$, then there exists a neighbourhood V around y, such that $V \subset f(X)$. Then, $f(X)$ is an open subset of Y. Again, as $f(X)$ is the image of a compact set, is compact too. As Y is a Hausdorff space and $f(X)$ is a compact subset of Y, so $f(X)$ is a closed subset of Y. This gives $f(X)$ is both open and closed subset of Y and Y is connected, so $f(X) = Y$. Thus, f is surjective.

Q.3a Let

$$f : \mathbb{S}^1 \to \mathbb{S}^1$$
$$f(x) = -x$$

be the antipodal map. Show that it is homotopic to the identity.

Ans. Let $F(s, t) : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^1$ be a map defined by $F(s, t) = s.e^{\pi i t}$. This is a homotopy from the identity to the antipodal map.

Q.3b Show that if k is odd the same is true for the antipodal map $\mathbb{S}^k \to \mathbb{S}^k$.

Ans. We can write a point of $\mathbb{S}^k$ as $(z_1, \cdots, z_n)$ where $2n = k+1$ and each z_i is a complex number. Then we take the homotopy $\mathbb{S}^k \times [0, 1] \to \mathbb{S}^k$ defined by $F((z_1, \cdots, z_n), t) = (z_1, \cdots, z_n).e^{\pi i t}$. This is a homotopy from the identity map to the antipodal map.

Q.4 Show that $[0, 1] \times [0, 1]$ is not a manifold with boundary.

Ans. Let $X = [0, 1] \times [0, 1]$. Take the point $(0, 0) \in X$. Take a neighbouhood U of $(0, 0)$ in X. Now, if X is a manifold with boundary, then U is diffeomorphic to an open set V of $\mathbb{H}^2 = \{(x, y) : y \geq 0\}$. Let the diffeomorphism be f.
If f sends $(0, 0)$ to an interior point x of $\mathbb{H}^2$, then we shall get an open ball B as a nbhd of x and a simply-connected open set $U_1 \subset f^{-1}(B)$ around $(0, 0)$ such that U_1 is mapped into B under f. But, as f sends $(0, 0)$ to x, $f : U_1 \to f(U_1)$ is not a homeomorphism (as if we delete $(0, 0)$ and x from both sides, we shall get simply-connected set on one side and not simply-connected set on the other side). So, $(0, 0)$ can't be sent to an interior point of $\mathbb{H}^2$ under f.

The same arguement shows that under any diffeomorphism $f : U \to V$, the boundary goes into the boundary of $\mathbb{H}^2$. As $f : U \to V$ is a smooth map, there is an open set $U_2 \subset \mathbb{R}^2$ containing U and a smooth map $F : U_2 \to \mathbb{R}^2$ so that the restriction of F to U is f. As f is a diffeomorphism, there is a smooth map $g : V \to U$, so there is an open set V_2 in $\mathbb{R}^2$ containing V and a smooth map $G : V_2 \to \mathbb{R}^2$ and the restriction of G to V is g. Also, $G \circ F$ on U is the identity.
The tangent space of U_2 at $(0, 0)$ has two linearly independent vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. As $G \circ F$ is identity on U, $d(G \circ F)$ is identity on these two vectors, so the image of these vectors under dF is again linearly independent.
But, as F sends the boundary of U to the boundary of $\mathbb{H}^2$, thus the image of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ under dF is linearly dependent, which is a contradiction to the fact proven in the last paragraph.
Therefore, $X = [0, 1] \times [0, 1]$ is not a manifold with boundary.

Q.5 Show that the Brouwer Fixed Point Theorem is false for the open ball of radius $a > 0$

$$\mathbb{B}^k = \{x \in \mathbb{R}^k | \parallel x \parallel < a\}.$$

Ans. Consider the following map $g : \mathbb{B}^k \to \mathbb{R}^k$ by

$$g(x) = 2/3.\{x + (a/2, 0, \cdots, 0)\}.$$

Here

$$\parallel 2/3.\{x + (a/2, 0, \cdots, 0)\} \parallel \leq 2/3.\{\parallel x \parallel + a/2\} < 2/3.(a + a/2) = a.$$

Therefore, the image of g is in $\mathbb{B}^k$. Again, g is fixed point free as $g(x) = x$ implies $x = (a, 0, \cdots, 0)$.